values of the parameters f_0 and Δp_* often encountered.

The self-similar solutions found permit an estimate of the influence of compressibility of the ground during unloading on the motion parameters even when there is no self-similarity. They can be used to verify the efficiency of the approximate methods of solution.

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THE RELATION BETWEEN MATHEMATICAL EXPECTATIONS OF STRESS AND STRAIN TENSORS IN STATISTICALLY ISOTROPIC HOMOGENEOUS ELASTIC BODIES

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The subject of this paper is the investigation of elastic solid bodies which conform to Hooke's law

$$\sigma_{ij} = c_{ijmn}\varepsilon_{mn} \tag{0.1}$$

Here the tensor of elastic moduli c_{ijmn} is considered to be a stationary random function of coordinates x_k with isotropic mathematical expectation

$$\langle c_{ijmn}\rangle = \lambda^\circ\delta_{ij}\delta_{mn} + \mu^\circ(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}) \tag{0.2}$$

where λ° and μ° are invariant physical constants, δ_{ij} is Kronecker's tensor.

Among such bodies are for example (in the region of small deformations) polycrystals without predominant directions of anisotropy and quasi-isotropic composite bodies.

At the present time the case of macroscopically homogeneous deformation of statistically isotropic homogeneous bodies has been studied in detail in [1–4] and others. Here the relationship between the mathematical expectations of stress and strain tensors can be written in the form

$$\langle\sigma_{ij}\rangle = 2\mu\langle\varepsilon_{ij}\rangle + \lambda\langle\varepsilon_{kk}\rangle\delta_{ij} \tag{0.3}$$

where μ and λ are "effective" Lamé's constants and do not coincide with μ° and λ°. The constants mentioned can be calculated from given statistical characteristics of the stationary random tensor c_{ijmn} by solving the three-dimensional nonlinear stochastic problem. An appropriate solution in the first approximation was obtained in [1]. Most general and reliable results in this area can be found in [4]. The general case where the fields of tensors σ_{ij} and ε_{ij} are nonstationary is more difficult. It was examined in [5] where, however, the arbitrary assumption was used that the centered part of the random vector of displacements of points of the body can be represented in the form

$$u_i^* = u_i - \langle u_i \rangle = a_{ik} \langle u_k \rangle \tag{0.4}$$

where a_{ik} is the random tensor the value of which is taken from the solution of the stationary problem for a body with identical statistical properties.

The nonstationary problem is examined below without any hypotheses. The solution is constructed in the form of a series which satisfies the equations of equilibrium for a volume element of the body and the equations of continuity of strain. The coefficients of this series are stationary tensors which are independent of the shape of the body and independent of external forces acting on it. The coefficients are characteristics of elastic properties of the body and are determined according to the given stationary random tensor c_{ijmn}.

From the constructed solution, relationships follow between mathematical expectations of stress and strain tensors. The relationships have a form which is analogous to the relationships between stresses and strains in the multicouple stress theory of elasticity ([6, 7] and others). The solution also gives the differential equations for mathematical expectation of the strain tensor. In addition, this paper presents a statistical interpretation of the couple and multicouple stress theory of elasticity as it applies to the quasi-isotropic elastic bodies. An algorithm is outlined for the computation of physical constants, which enter into this theory, from a given random field of the tensor of elastic moduli. The fundamental idea of this work was already expressed in [8], where the series (2.1) presented below and the recursion systems (4.6), (4.7) and (4.8) were examined. However, in writing the cited reference [8], the author did not yet have at his disposal concrete computations and was forced to resort to considerations of intuitive nature. This led to a mistake, the essence of which consists in the unfortunate selection of the deterministic tensor $\varepsilon_{ij}^{\vee}$ which was used for the expansion of the solution. Further investigation showed that consistency of the proposed method is ensured only under the condition that $\varepsilon_{ij}^{\vee}$ is taken as the mathematical expectation of the strain tensor, i.e. it is necessary to assume that $\varepsilon_{ij}^{\vee} = \langle \varepsilon_{ij} \rangle$. Results presented below correct the work of [8] in the direction indicated.

1. The problem to be examined reduces to the solution of the following equations in the linear theory of elasticity

$$\begin{gathered} \partial_j \sigma_{ij} + F_i = \partial_j c_{ijmn} \varepsilon_{mn} + F_i = 0 \\ \epsilon^{imk} \epsilon^{jnl} \frac{\partial^2 \varepsilon_{mn}}{\partial x_k \partial x_l} = 0 \\ \sigma_{ij} N_j = c_{ijmn} \varepsilon_{mn} N_j = f_i \ (\text{on } \Omega) \end{gathered} \tag{1.1}$$

Here x_s are orthogonal Cartesian coordinates, ε_{ij} are components of the strain tensor, σ_{ij} are components of the stress tensor, c_{ijmn} is the tensor of elastic moduli, Ω is the surface bounding the body, N_j is the unit vector of the external normal to this surface, F_i and f_i are external body and surface forces, ϵ^{imk} is the unitary antisymmetric pseudotensor of Levi-Civita

$$\epsilon^{imk} = \begin{cases} \pm 1 \\ 0 \end{cases} \tag{1.2}$$

which acquires the upper value when all its indices are different and the lower value when at least one pair of its indices is the same. The plus sign in (1.2) must be taken when

$$i,\ m,\ k = 1,2,3,\ \sim 3,1,2 \sim 2,3,1$$

the minus sign for all other sequences of indices.

The product of such tensors which enter into (1.1) can be expressed through Kronecker's unitary tensors, namely

$$\epsilon^{imk}\epsilon^{jnl} = \delta_{ij}\delta_{mn}\delta_{kl} - \delta_{mj}\delta_{in}\delta_{kl} + \delta_{jm}\delta_{kn}\delta_{il} - \\ - \delta_{jk}\delta_{mn}\delta_{il} + \delta_{jk}\delta_{in}\delta_{ml} - \delta_{ij}\delta_{kn}\delta_{lm} \tag{1.3}$$

In previous equations, and also in all subsequent equations, the convention of summation over repeated ("dummy") indices is utilized.

The tensor of elastic moduli is considered to be a known stationary random function of coordinates.

It is assumed that its mathematical expectation $\langle c_{ijmn}\rangle$ (0.1) is given and the correlation tensor of the order eight has the form

$$C_{ijmn}^{klpq}(M_1M_2) = \langle \overset{\bullet}{c}_{ijmn}(M_1)\, \overset{\bullet}{c}_{klpq}(M_2)\rangle \tag{1.4}$$

where M_1 and M_2 are two arbitrary points of the body, while

$$\overset{\bullet}{c}_{ijmn}(M) = c_{ijmn}(M) - \langle c_{ijmn}(M)\rangle \tag{1.5}$$

In view of the stationary character of c_{ijmn} the moment of relationship (1.4) will be a function only of

$$\xi_s = x_s^{(1)} - x_s^{(2)} \tag{1.6}$$

where

$$C_{ijmn}^{klpq}(\xi_s) = C_{klpq}^{ijmn}(-\xi_s) \tag{1.7}$$

The most general form of correlation tensor $C_{ijmn}^{klpq}(\xi_s)$ is presented in [3].

External forces F_i and f_i acting on the body are subsequently considered as determined quantities.

2. Following the method proposed in [8], we shall seek a random strain tensor ε_{ij} in the form of an expansion

$$\varepsilon_{ij} = \varepsilon_{ij}\check{} + \alpha_{ijmn}\,\check{\varepsilon}_{mn} + \beta_{ijmnp}\frac{\partial \check{\varepsilon}_{mn}}{\partial x_p} + \gamma_{ijmnpq}\frac{\partial^2 \check{\varepsilon}_{mn}}{\partial x_p\, \partial x_q} + \ldots \tag{2.1}$$

here $\varepsilon_{ij}\check{}\ (x_k)$ is a deterministic symmetric tensor of order two. Tensors of order four α_{ijmn}, of order five β_{ijmnp}, of order six γ_{ijmnpq}, etc. are random functions of coordinates x_k.

The idea of the method consists in the selection of a deterministic tensor ε_{ij} in such a manner that the coefficients of series (2.1) will be stationary random functions which are completely determined by the given random tensor c_{ijmn} and do not depend on the shape of the body, on its dimensions, and on external forces which act on it.

This idea can be realized if it is assumed that the coefficients of the series (2.1) are centered random functions, i.e.

$$\langle \alpha_{ijmn}\rangle = \langle \beta_{ijmnp}\rangle = \langle \gamma_{ijmnpq}\rangle = \ldots = 0 \tag{2.2}$$

In this case the deterministic tensor $\varepsilon_{ij}\check{}$ turns out to be identical with the mathematical expectation of the strain tensor

$$\varepsilon_{ij}\check{} = \langle \varepsilon_{ij}\rangle \tag{2.3}$$

Here, previously and subsequently, the symbol $\langle\ \ \rangle$ indicates averaging over the entire representation.

Taking advantage of (0.1) and (2.1) we have the following series for the stress tensor:

$$\sigma_{ij} = A_{ijmn}\check{\varepsilon}_{mn} + B_{ijmnp}\frac{\partial \check{\varepsilon}_{mn}}{\partial x_p} + C_{ijmnpq}\frac{\partial^2 \check{\varepsilon}_{mn}}{\partial x_p \partial x_q} + \ldots \tag{2.4}$$

where

$$A_{ijmn} = c_{ijmn} + c_{ijrs}\alpha_{rsmn}, \quad B_{ijmnp} = C_{ijrs}\beta_{rsmnp}, \quad C_{ijmnpq} = C_{ijrs}\gamma_{rsmnpq} \tag{2.5}$$

Coefficients of series (2.4) are not centered quantities, which corresponds to

$$\langle\sigma_{ij}\rangle = \langle A_{ijmn}\rangle\check{\varepsilon}_{mn} + \langle B_{ijmnp}\rangle\frac{\partial \check{\varepsilon}_{mn}}{\partial x_p} + \langle C_{ijmnpq}\rangle\frac{\partial^2 \check{\varepsilon}_{mn}}{\partial x_p \partial x_q} + \ldots \tag{2.6}$$

$$\sigma_{ij}^* = \sigma_{ij} - \langle\sigma_{ij}\rangle = \overset{*}{A}_{ijmn}\check{\varepsilon}_{mn} + \overset{*}{B}_{ijmnp}\frac{\partial \check{\varepsilon}_{mn}}{\partial x_p} + \overset{*}{C}_{ijmnpq}\frac{\partial^2 \check{\varepsilon}_{mn}}{\partial x_p \partial x_q} + \ldots \tag{2.7}$$

where

$$\overset{*}{A}_{ijmn} = A_{ijmn} - \langle A_{ijmn}\rangle, \qquad \overset{*}{B}_{ijmnp} = B_{ijmnp} - \langle B_{ijmnp}\rangle \tag{2.8}$$

etc.

Substituting into the system of equations (1.1)

$$\sigma_{ij} = \langle\sigma_{ij}\rangle + \sigma_{ij}^*, \quad \varepsilon_{ij} = \langle\varepsilon_{ij}\rangle + \varepsilon_{ij}^* \tag{2.9}$$

we arrive at two systems

$$\frac{\partial\langle\sigma_{ij}\rangle}{\partial x_j} + F_i = 0, \quad \epsilon^{imk}\epsilon^{jnl}\frac{\partial^2 \check{\varepsilon}_{mn}}{\partial x_k \partial x_l} = 0, \quad \langle\sigma_{ij}\rangle N_j = f_i \quad (\text{on } \Omega) \tag{2.10}$$

$$\frac{\partial \overset{*}{\sigma}_{ij}}{\partial x_j} = 0, \quad \epsilon^{imk}\epsilon^{jnl}\frac{\partial^2 \overset{*}{\varepsilon}_{mn}}{\partial x_k \partial x_l} = 0, \quad \overset{*}{\sigma}_{ij}N_j = 0 \quad (\text{on } \Omega) \tag{2.11}$$

Let us examine them consecutively.

3. As we specified above, the coefficients of series (2.1) represent stationary random functions of coordinates. In this case the coefficients of series (2.4) will also be stationary random functions. These coefficients are determined by Eqs. (2.5) as products of stationary random functions. It is apparent that the mathematical expectations of these coefficients must be isotropic tensors, because otherwise the relationship between the mathematical expectations of stress and strain tensors would have an anisotropic character which would be a contradiction of the assumption of statistical isotropy of the body which is being examined.

The most general form of an anisotropic fixed tensor of the fourth order is given by the following expression:

$$\langle A_{ijmn}\rangle = a_1\delta_{ij}\delta_{mn} + a_2(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}) \tag{3.1}$$

where a_1 and a_2 are scalar constants. Tensors of fifth order (and generally uneven orders) cannot be isotropic. In accordance with this we have

$$\langle B_{ijmnp}\rangle = 0 \tag{3.2}$$

The most general form of an isotropic fixed tensor of the sixth order is given by the following expression:

$$\begin{aligned}\langle C_{ijmnpq}\rangle &= g_1\delta_{ij}\delta_{mn}\delta_{pq} + {}^1/_2\, g_2(\delta_{mp}\delta_{nq} + \delta_{np}\delta_{mq})\delta_{ij} + \\ &+ {}^1/_2\, g_3(\delta_{ip}\delta_{jq} + \delta_{jp}\delta_{iq})\delta_{mn} + {}^1/_2\, g_4(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in})\delta_{pq} + \\ &+ {}^1/_4\, g_5[(\delta_{jp}\delta_{nq} + \delta_{np}\delta_{jq})\delta_{im} + (\delta_{ip}\delta_{nq} + \delta_{in}\delta_{pq})\delta_{jm} + \\ &+ (\delta_{jp}\delta_{mq} + \delta_{mp}\delta_{jq})\delta_{in} + (\delta_{ip}\delta_{mq} + \delta_{mi}\delta_{pq})\delta_{jn}]\end{aligned} \tag{3.3}$$

where g_1, g_2, g_3, g_4 and g_5 are scalar constants. In (3.1) and (3.3) it is taken into consideration that the investigated tensors, as is evident from (2.6), must permit the transposition of index pairs (i, j), (m, n) and (p, q).

Substituting (3.1), (3.2) and (3.3) into (2.6) we obtain

$$\langle\sigma_{ij}\rangle = 2a_2 \check{\varepsilon}_{ij} + a_1\check{\varepsilon}_{kk}\delta_{ij} + g_1\Delta\check{\varepsilon}_{kk}\delta_{ij} + g_2\frac{\partial^2\check{\varepsilon}_{kl}}{\partial x_k\,\partial x_l}\delta_{ij} + g_3\frac{\partial^2\check{\varepsilon}_{kk}}{\partial x_i\,\partial x_j} + \\ + g_4\Delta\check{\varepsilon}_{ij} + g_5\left(\frac{\partial^2\check{\varepsilon}_{ik}}{\partial x_j\,\partial x_k} + \frac{\partial^2\check{\varepsilon}_{jk}}{\partial x_i\,\partial x_k}\right) + \dots \tag{3.4}$$

This expression can be transformed somewhat if we take into account that the tensor $\check{\varepsilon}_{ij}$ is subject to Saint-Venant's relationships $(2.10)_2$, which with utilization of (1.3) are reduced to the following form:

$$\Delta\check{\varepsilon}_{ij} + \frac{\partial^2\check{\varepsilon}_{kk}}{\partial x_i\,\partial x_j} - \frac{\partial^2\check{\varepsilon}_{ik}}{\partial x_j\,\partial x_k} - \frac{\partial^2\check{\varepsilon}_{jk}}{\partial x_i\,\partial x_k} = 0 \tag{3.5}$$

Into (3.4) and (3.5) the following common notation is introduced:

$$\Delta(\ ) = \frac{\partial^2(\ )}{\partial x_k\,\partial x_k} \tag{3.6}$$

It follows from (3.5) that

$$\Delta\check{\varepsilon}_{kk} = \frac{\partial^2\check{\varepsilon}_{kl}}{\partial x_k\,\partial x_l}, \qquad \frac{\partial^2\check{\varepsilon}_{ik}}{\partial x_j\,\partial x_k} + \frac{\partial^2\check{\varepsilon}_{kj}}{\partial x_i\,\partial x_k} = \Delta\check{\varepsilon}_{ij} + \frac{\partial^2\check{\varepsilon}_{kk}}{\partial x_i\,\partial x_j} \tag{3.7}$$

Equations (3.7) permit to reduce (3.4) to a simpler form

$$\langle\sigma_{ij}\rangle = 2a_2\check{\varepsilon}_{ij} + a_1\check{\varepsilon}_{kk}\delta_{ij} + b_1\frac{\partial^2\check{\varepsilon}_{kk}}{\partial x_i\,\partial x_j} + b_2\Delta\check{\varepsilon}_{ij} + b_3\Delta\check{\varepsilon}_{kk}\delta_{ij} + \dots \tag{3.8}$$

where

$$\begin{aligned} b_1 &= g_3 + g_5 \\ b_2 &= g_4 + g_5 \\ b_3 &= g_1 + g_2 \end{aligned} \tag{3.9}$$

From the fact that the mathematical expectation of the strain tensor $\langle\varepsilon_{ij}\rangle = \check{\varepsilon}_{ij}$ is subject to Saint-Venant's relationships it follows that

$$\check{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \check{u}_i}{\partial x_j} + \frac{\partial \check{u}_j}{\partial x_i}\right) \tag{3.10}$$

where the vector $\check{u}_s$ is the mathematical expectation of the displacement vector.

Substituting (3.10) into (3.8) and then introducing (3.8) into (2.10) we arrive at a system of three equations in partial derivatives with three unknown $\check{u}_s$.

This system (taking into consideration in the series (3.8) all the terms) will have an infinitely high order. If, however, in the series (3.8) only those terms are retained which were written out above, then a system is obtained which is of an order twice higher than the system of equations in the classical theory of elasticity. This last system will be identical in its form to the system of equations of the couple stress theory of elasticity ([1, 7] and others).

So far here it remains unknown (1) how to determine the scalar constants a_1, a_2, $g_1, \dots, g_5$ which enter into series (3.4), from given statistical characteristics of the random field of the tensor of elastic moduli c_{ijmn}; (2) what boundary conditions must be placed on the surface which bounds the body so that the problem will become completely determined? The substitution of series (3.4) into $(2.10)_3$ gives only three boundary conditions which is apparently insufficient.

For an answer to these two questions it is necessary to examine the system (2.11).

4. Let us substitute the series (2.7) into Eq. $(2.11)_1$ and require that this equation be satisfied by equating to zero coefficients of the tensor $\varepsilon_{ij}^{\vee}$ and all its partial derivatives. An analogous procedure is used with the equation of continuity $(2.11)_2$, substituting into it the series (2.1). As a result the following sequence of equations is obtained

$$\frac{\partial \overset{*}{A}_{ijmn}}{\partial x_j} = 0, \qquad \epsilon^{irk}\epsilon^{jsl}\frac{\partial^2 \alpha_{rsmn}}{\partial x_k \partial x_l} = 0 \tag{4.1}$$

$$\frac{\partial \overset{*}{B}_{ijmnp}}{\partial x_j} = -\overset{*}{A}_{ipmn}, \qquad \epsilon^{irk}\epsilon^{jsl}\frac{\partial^2 \beta_{rsmnp}}{\partial x_k \partial x_l} = \Phi_{ijmnp} \tag{4.2}$$

$$\frac{\partial \overset{*}{C}_{ijmnpq}}{\partial x_j} = -\overset{*}{B}_{iqmnp} - \overset{*}{B}_{ipmnq}, \quad \epsilon^{irk}\epsilon^{jsl}\frac{\partial^2 \gamma_{ijmnpq}}{\partial x_k \partial x_l} = \Psi_{ijmnpq} \tag{4.3}$$

in which

$$\Phi_{ijmnp} = -\left[\epsilon^{irk}\epsilon^{jsp} + \epsilon^{irp}\epsilon^{jsk}\right]\frac{\partial \alpha_{rsmn}}{\partial x_k} \tag{4.4}$$

$$\begin{aligned}\Psi_{ijmnpq} = &-\frac{1}{2}\left[\epsilon^{irq}\epsilon^{jsk} + \epsilon^{irk}\epsilon^{isq}\right]\frac{\partial \beta_{rsmnp}}{\partial x_k} - \\ &-\frac{1}{2}\left[\epsilon^{irp}\epsilon^{jsk} + \epsilon^{irk}\epsilon^{jsp}\right]\frac{\partial \beta_{rsmnq}}{\partial x_k} - \\ &-\frac{1}{2}\left[\epsilon^{irp}\epsilon^{jsq} + \epsilon^{irq}\epsilon^{jsp}\right]\alpha_{rsmn}\end{aligned} \tag{4.5}$$

In writing (4.4) and (4.5) in the expanded form it is necessary to take into account (1.3).

It was stipulated above that the coefficients of series (2.1) are considered centered stationary functions of coordinates, and it was pointed out that in this case the coefficients of the series for the stress tensor (2.4) also are stationary random functions.

Utilizing (2.5) and taking into consideration the above remark, we can reduce the system of equations (4.1), (4.2) and (4.3) to the form

$$\frac{\partial c_{ijrs}\alpha_{rsmn}}{\partial x_j} = -\frac{\partial c_{ijmn}}{\partial x_j}, \qquad \epsilon^{irk}\epsilon^{jsl}\frac{\partial^2 \alpha_{rsmn}}{\partial x_k \partial x_l} = 0 \tag{4.6}$$

$$\frac{\partial c_{ijrs}\beta_{rsmnp}}{\partial x_j} = -\overset{*}{c}_{ipmn} - c_{iprs}\alpha_{rsmn} + \langle c_{iprs}\alpha_{rsmn}\rangle \tag{4.7}$$

$$\epsilon^{irk}\epsilon^{jsl}\frac{\partial^2 \beta_{rsmnp}}{\partial x_k \partial x_l} = \Phi_{ijmnp}$$

$$\begin{aligned}\frac{\partial c_{ijrs}\gamma_{rsmnpq}}{\partial x_j} = &-\frac{1}{2}c_{iqrs}\beta_{rsmnp} - \frac{1}{2}c_{iprs}\beta_{rsmnq} + \\ &+\frac{1}{2}\langle c_{iqrs}\beta_{rsmnp}\rangle + \frac{1}{2}\langle c_{iprs}\beta_{rsmnq}\rangle\end{aligned}$$

$$\epsilon^{irk}\epsilon^{jsl}\frac{\partial^2 \gamma_{rsmnpq}}{\partial x_k \partial x_l} = \Psi_{ijmnpq} \tag{4.8}$$

The system (4.6) decomposes into six independent systems of equations (according to the number of different combinations of indices m and n). Each of these is identical with the system of equations in the linear theory of elasticity. The role of the strain tensor in these systems is played by the tensor $\alpha_{ij(mn)}$. The tensor of elastic moduli is c_{ijrs}, while external body forces are given by the equation

$$F_i = \frac{\partial c_{ij(mn)}}{\partial x_j} \tag{4.9}$$

The requirement for the tensor α_{ijmn} to be stationary (in some respects analogous to the requirement of periodicity of the solution) replaces the boundary conditions for the indicated six systems of differential equations.

Taking into consideration this statement, the problem of determining the tensor is completely solved by means of integrating six systems (4.6).

In essence this particular problem was examined in [1, 2 and 4], which were devoted to the investigation of the relationship between the mathematical expectations of tensors ε_{ij} and σ_{ij} for polycrystals in the particular case when

$$\langle \varepsilon_{ij} \rangle = \text{const}, \qquad \langle \sigma_{ij} \rangle = \text{const}$$

After determination of α_{ijmn} it becomes also possible to find the next coefficient of the series (2.1), i.e. β_{ijmnp}. For this purpose it is necessary to take advantage of system (4.7) which decomposes into 18 independent systems of equations (according to the number of different combinations of indices m, n and p). Each of the indicated systems is identical to equations of the linear theory of elasticity in the presence of internal stresses in the body. The role of the strain tensor in these equations is played by the tensor $\beta_{ij(mnp)}$, the tensor of elastic moduli is c_{ijmn}. External body forces are determined by the expression

$$F_{i(mnp)} = -\overset{*}{c}_{i(pmn)} - c_{i(p)rs}\,\alpha_{rs(mn)} + \langle c_{i(p)rs}\alpha_{rs(mn)} \rangle \tag{4.10}$$

The incompatibility tensor is given by Eq. (4.4).

The requirement for the tensor β_{ijmnp} to be stationary is equivalent to specifying boundary conditions for the 18 systems of differential equations indicated above.

In an analogous manner the system (4.8), and also all subsequent systems of equations for coefficients of the series (2.1), decompose into independent systems of equations, each of which is identical to equations of linear theory of elasticity for a body with internal stresses. In this connection the role of boundary conditions for all these systems of equations is played by the requirement of stationary behavior of their solutions. By the same token all coefficients of series (2.1) can be determined successively by solving the recurrent system of partial differential equations described above.

5. As a result of operations described above, two series are constructed, (2.1) and (2.4). The first series expresses the strain tensor. The second series represents the stress tensor through the mathematical expectation of the strain tensor and through partial derivatives of the mathematical expectation with respect to coordinates. These series have the following properties:

the coefficients of these series are stationary random functions of coordinates, independent of the shape of the body, its dimensions, and external forces acting on it. The latter are considered in this connection as determined vectors;

the series (2.4) identically satisfies the system of equations $(2.11)_1$;

the series (2.1) identically satisfies the system of equations $(2.11)_2$;

Series (3.8), which expresses the mathematical expectation of the stress tensor through the mathematical expectation of the strain tensor, is known to the same extent as series (2.4).

By the same token the above discussion exhausts the question about the relationship between averaged stresses and strains in statistically isotropic, homogeneous, linearly elastic bodies.

However, the method of calculation of mathematical expectation of the strain tensor $\langle\varepsilon_{ij}\rangle = \varepsilon_{ij}\check{}$ has not been completely explained yet in each concrete particular case (i.e. as a function of body dimensions, its shape, and external forces acting upon it).

For this tensor a system of differential equations $(2.10)_{1,2}$ of infinitely high order was obtained above. For this system so far there are only three boundary conditions $(2.10)_3$ which are insufficient for single-valued determination of $\varepsilon_{ij}\check{}$. The missing boundary conditions can be formulated if it is taken into consideration that so far the equation $(2.11)_3$, which determines the boundary value of the centered random vector $\sigma_{ij}{}^*N_j$, has not been utilized yet.

From the last equation we can obtain an infinite set of boundary conditions for the deterministic tensor $\varepsilon_{ij}\check{}$ and its partial derivatives with respect to coordinates. For example, these conditions can be derived by equating to zero the moments of components of random vector $\sigma_{ij}{}^*N_j$ on the surface bounding the body. Such a method is most obvious; however, it results in nonlinear boundary conditions.

An alternate possibility is to derive the boundary conditions for the tensor $\varepsilon_{ij}{}^*$ and its partial derivatives from the variational problem formulated in such a manner that its Euler's equations are identical with system $(2.10)_1$ written in terms of displacements u_s. This method results in linear boundary conditions. The system of equations obtained in this case for determination of mathematical expectation of the random vector of displacement of points of the body u_s is consistent with equations of the multi-couple stress theory of elasticity [6, 7].

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